Nonlinear chaotic dynamics of the chaotic laser diodes with an additional optical injection is computed within rate equations model, based on the set of rate equations for the slave laser electric complex amplitude and carrier density. To calculate the system dynamics in a chaotic regime the known chaos theory and non-linear analysis methods such as a correlation integral algorithm, the Lyapunov’s exponents and Kolmogorov entropy analysis are used. There are listed the data of computing dynamical and topological invariants such as the correlation, embedding and Kaplan-Yorke dimensions, Lyapunov’s exponents, Kolmogorov entropy etc. New data on topological and dynamical invariants are computed and firstly presented.

1. Introduction

The elements of chaotic dynamics in different laser systems and devices, including semiconductor lasers, laser diodes, resonators etc are of a great importance and interest because of their potential applications in laser physics and quantum electronics, optical secure communications and cryptography, and many others. At the same time, the laser’s relaxation oscillation limits the bandwidth of chaotic light emitted from a laser diode and similar devices with single optical injection or feedback. This circumstance as well as a general interest to new theoretical dynamics phenomena make necessary the further studying and improvement the main features of the optical chaos communications.

From the other side, there is a general interest to studying the chaotic laser systems provided a necessity of the further development of a general theory of dynamic systems and a chaos.

Let us remind that according to Refs. [1-15], under the definite conditions, such systems are described by the corresponding model, when Hamiltonians are possessing only a few degrees of freedom. For the low-dimensional chaotic case the corresponding conditions of transition to deterministic chaos in the system dynamics are quite well understood at the classical level [1-4].

Under quantum treatment of the problem, the similar systems (in particular, the diatomic molecules in a resonant electromagnetic field) are studied with using the known quasiclassical approach [2]. At the theoretical level, the majority of studies, devoted to chaos phenomena in molecular dynamics, is carried out with the using simple tools of dynamical systems theory and qualitative theory of differential equations. New field of investigations of the quantum and other systems has been provided by the known progress in a development of a nonlinear analysis and chaos theory methods [1-12,17-30].

In Refs. [11,27-33] the authors applied different approaches to quantitative studying regular and chaotic dynamics of atomic and molecular systems interacting with a strong electromagnetic field and laser systems.

The most popular approach to studying nonlinear dynamics of chaotic systems includes the combined using the advanced nonlinear analysis and a chaos theory methods such as the autocorrelation function method, multi-fractal formalism, mutual information approach, correlation integral analysis, false nearest neighbour algorithm, Lyapunov exponent’s analysis, surrogate data method, stochastic propagators method, memory and Green’s functions approaches etc (see details in Refs. [17-24]).
In Ref. [1] the authors experimentally and numerically demonstrate the route to band width enhanced chaos in a chaotic laser diode with an additional optical injection; they used the own unique experimental setup, which includes a distributed feedback (DFB) laser with a 4 m fiber ring feedback cavity (the slave laser) and the other solitary DFB laser as an injection laser (the master laser) to enlarge the bandwidth of the chaotic laser (see detailed description in Ref. [1]). The concrete technological characteristics are as follows: slave laser is biased at 28.0 mA (1.27 times threshold), and its wavelength is stabilized at 1553.8 nm with 0.3 nm linewidth (at −20 dB) and a 35 dB side mode suppression ratio; respectively, the laser’s output power is 0.7 mW, and the relaxation frequency and modulation bandwidth were about 2 GHz and 5 GHz. The original set of the chaotic states before optical injection is obtained with −6.1 dB optical feedback (the feedback injection strength with a scale of the solitary slave laser’s power).

In this paper we present the corresponding results of computing the characteristic dynamical and topological invariants of the nonlinear dynamics of the chaotic laser diode with an additional optical injection (all characteristics are corresponding to parameters of the Ref. [1]).

2. Chaos-geometric approach to dynamics of the chaotic systems

As the main ideas of the quantum-dynamic approach to diatomic molecule in an electromagnetic field are in details presented in the Refs. [5-7,2], here we will restrict yourself only by some necessary elements.

In order to perform the detailed analysis of the chaotic regime polarization time series, further a total dynamics of the quantum system in an electromagnetic field and to calculate the fundamental topological and dynamical invariants of the system in a chaotic regime we used the universal chaos-geometric approach, presented earlier (see, c.g., [5-7,19-20]).

Generally speaking, the approach includes a set of such non-linear analysis and a chaos theory methods as the correlation integral approach, multi-fractal and wavelet analysis, average mutual information, surrogate data, Lyapunov’s exponents and Kolmogorov entropy approach, spectral methods, nonlinear prediction (predicted trajectories, neural network etc) algorithms.

One of the important tasks here is to determine the corresponding embedding dimension and to reconstruct a Euclidean space $R^d$ large enough so that the set of points $d_A$ can be unfolded without ambiguity. In accordance with the embedding theorem, the embedding dimension, $d_E$, must be greater, or at least equal, than a dimension of attractor, $d_A$, i.e. $d_E \geq d_A$.

Usually one should use several standard approaches to reconstruction of the attractor dimension (see, e.g., [17-20]). The correlation integral analysis is one of the widely used techniques to investigate the signatures of chaos in a time series. The analysis uses the correlation integral, $C(r)$, to distinguish between chaotic and stochastic systems.

To compute the correlation integral, the algorithm of Grassberger and Procaccia is the most commonly used approach. According to this algorithm, the correlation integral is

$$C(r) = \lim_{N \to \infty} \frac{2}{N(N-1)} \sum_{i \neq j \in [N]} H(r - \|y_i - y_j\|)$$

where $H$ is the Heaviside step function with $H(u) = 1$ for $u > 0$ and $H(u) = 0$ for $u \leq 0$, $r$ is the radius of sphere centered on $y_i$ or $y_j$, and $N$ is the number of data measurements.

In order to perform the verification of the results obtained by means of the correlation integral analysis, one could use so called known surrogate data method. This approach makes use of the substitute data generated in accordance to the probabilistic structure underlying the original data.

The important dynamical invariants of a chaotic system are the Lyapunov’s exponents (see, c.g., [17-20]). These characteristics can be defined as asymptotic average rates, they are independent of the initial conditions, and
Therefore they do comprise an invariant measure of attractor. Saying simply, the Lyapunov’s exponents are the parameters to detect whether the system is chaotic or not.

Another important characteristics, namely, the Kolmogorov entropy \( K_{\text{ent}} \) measures the average rate at which information about the state is lost with time. According to the definition, the Kolmogorov entropy can be determined as the sum of the positive Lyapunov’s exponents.

The estimate of the dimension of the attractor is provided by the Kaplan and York conjecture:

\[
d_L = j + \frac{\sum \lambda_\alpha}{|\sum \lambda_{j=1}|},
\]

where \( j \) is such that \( \sum \lambda_\alpha > 0 \) and \( \sum \lambda_\alpha < 0 \), and the Lyapunov’s exponents \( \lambda_\alpha \) are taken in descending order.

There are a few approaches to computing the Lyapunov’s exponents. One of them computes the whole spectrum and is based on the Jacobi matrix of system. In this work we use an advanced algorithm with fitted map with higher order polynomials. To calculate the spectrum of the Lyapunov’s exponents, one could determine the time delay \( \tau \) and embed the data in the four-dimensional space. In this point it is very important to determine the Kaplan-York dimension and compare it with the correlation dimension, defined by the Grassberger-Proccacia algorithm.

As a rule, the calculation results of the state-space reconstruction are highly sensitive to the length of data set (i.e. it must be sufficiently large) as well as to the time lag and embedding dimension correctly determined.

Indeed, there are limitations on the applicability of chaos theory for observed (finite) dynamical variable series arising from the basic assumptions that these series must be infinite. A finite and small data set may probably result in an underestimation of the actual dimension of the process. The details of the computational procedures and algorithms can be also found in Refs. [27-46].

3. Nonlinear dynamics of the chaotic laser diode: Some results and conclusions

Below we present the results of of computing the dynamical and topological invariants of the nonlinear dynamics of the chaotic laser diode system with an additional optical injection. According to [1], the dynamics of this system can be described by a set of rate equations for the slave laser electric complex amplitude \( F \) and carrier density \( n \), correspondingly and is represented as follows:

\[
\frac{dF}{dt} = \frac{1 + i\beta}{2} \left( \frac{g(n - n_0)}{1 + |F|^2} - \tau F \right) + \frac{k_i}{\tau_i} F(t - \tau) \cdot \exp[-i2\pi\eta\tau] + \frac{k_j}{\tau_j} F \cdot \exp[i\Delta\eta],
\]

\[
\frac{dn}{dt} = \frac{i}{qV} \left( \frac{n}{\tau_N} - \frac{g(n - n_0)}{1 + |F|^2} |F|^2 \right) + G(n)
\]

where \( k_i \) and \( k_j \) denote the feedback and injection strength, the amplitude of injection laser \( |F_j| \) is equal to that of the solitary slave laser, and \( \Delta\eta = \eta_j - \eta_s \) is the detuning between the injection and the slave lasers. The feedback delay \( \tau = 20 \) ns is set in the experimental setup [1]. As the input data for the solving the rate equations system the numerical values of the parameters have been used as follows (see more details in Ref. [1]): transparency carrier density \( n_0 = 0.455 \times 10^9 \) m\(^{-3} \), threshold current \( i_{\text{th}} \) = 22 mA, differential gain \( g = 1.414 \times 10^3 \) \( \mu \)m\(^3\) ns\(^{-1} \), the carrier lifetime \( \tau_N = 25 \) ns, photon lifetime \( \tau_p = 1.17 \) ps, round-trip time in laser intracavity \( \tau_c = 7.38 \) ps, the linewidth enhancement factor \( \beta = 5.0 \), gain saturation parameter \( \delta = 5 \times 10^{-3} \) \( \mu \)m\(^3\) and active layer volume \( V = 324 \) m\(^3\), the simulated slave laser is biased at \( 1.7i_{\text{th}} \) with 5.2 GHz modulation bandwidth.

According to data [1], under \( k_i = 0 \), a growth of the parameter \( k_j \) results in a period-doubling bifurcation route to chaos, followed
by a reversed route out of chaos. More exactly, a chaos is realized in the region about 0.04–0.16 of $k_f$ and bandwidths are about 4.0–6.2 GHz. The rate equations systems has been numerically solved and the corresponding time series for amplitude and density are obtained. The concrete step is an analysis of the corresponding time series with the $N = 10^4$ and $\Delta t = 2 \times 10^{-3} \text{ns}$. It is very important to declare that the dynamics of the chaotic laser diode system with an additional optical injection has the elements of a deterministic chaos (the strange attractor). In Table 1 we present the computational values of the correlation dimension $d_2$, the Kaplan-York attractor dimension ($d_L$), the Lyapunov’s exponents ($\lambda_i$), Kolmogorov entropy ($K_{\text{ent}}$), the Gottwald-Melbourne parameter.

### Table 1

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<th>$d_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<td>2.94</td>
<td>0.358</td>
<td>0.096</td>
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To conclude, the values of the dynamical and topological invariants (the correlation, Kaplan-York dimensions, the Lyapunov’s exponents etc) for the dynamics of the chaotic laser diode system with an additional optical injection are computed. In particular, the first two Lyapunov’s exponents are positive. These data indicate on emerging dynamical chaos elements in the laser diode system dynamics.

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DYNAMICAL AND TOPOLOGICAL INVARIANTS OF NONLINEAR DYNAMICS OF THE CHAOTIC LASER DIODES WITH AN ADDITIONAL OPTICAL INJECTION

Summary. Nonlinear chaotic dynamics of the of the chaotic laser diodes with an additional optical injection is computed within rate equations model, based on the a set of rate equations for the slave laser electric complex amplitude and carrier density. To calculate the system dynamics in a chaotic regime the known chaos theory and non-linear analysis methods such as a correlation integral algorithm, the Lyapunov’s exponents and Kolmogorov entropy analysis are used. There are listed the data of computing dynamical and topological invariants such as the correlation, embedding and Kaplan-Yorke dimensions, Lyapunov’s exponents, Kolmogorov entropy etc. New data on topological and dynamical invariants are computed and firstly presented.

Key words: Chaotic dynamics, laser diodes, dynamical and topological invariants